# A NEW LINEAR SCHEME FOR SOLVING THE 1-D BURGER'S EQUATION

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**Abstract.** A new model technique based on the linearization of Burger's equation is introduced. In this paper we consider the approximate solution of the following nonlinear one dimensional Burgers' equation. A new discretization scheme is introduced. A proof of convergence of the approximate solution is given and error estimates are derived. The numerical results obtained by the suggested technique are compared with the exact solution of the problem and also with other numerical methods. It is shown that our scheme is comparable with the others, and the numerical solution displays the expected convergence to the exact one as the mesh size is refined.

**Keywords.** One dimensional Burger's equation, convection-diffusion problems, time discretization methods, Rothe's functions.

## 1. Introduction

Burger [3] first introduced a nonlinear wave equation to investigate the wave propagation problem subject to linear dissipation and nonlinear advection terms. This equation is a non-linear parabolic one dimensional partial differential equation given by:

$$u_t + uu_x - \nu u_{xx} = f(x,t), \qquad (x,t) \in Q_T \tag{1.1}$$

Here $Q_T = \Omega \times I$, $\Omega \equiv (a,b)$, $\partial\Omega \equiv \{a,b\}$, $I \equiv (0,T)$, and $a$ and $b$ are real positive constants. We consider equation (1.1) associated with the Dirichlit boundary condition

$$u = 0, \qquad (x,t) \in \partial\Omega \times I, \tag{1.2}$$

and the initial condition

$$u(x,0) = u_0(x), \qquad x \in \Omega, \tag{1.3}$$

where $u(x,t)$ is the dependent variable, $x$ and $t$ are the independent variables, $\nu > 0$ is the kinematics viscosity of the fluid that controlling the balance between convection and diffusion, and $f$ and $u_0$ are given functions of their arguments.

The nonlinear convection diffusion equation (1.1) has attracted considerable interest during the last few decades, since it services as a mathematical model for wide range of applications, from the fluid dynamics and turbulence to shock wave formation and traffic flow ( cf. [1, 2, 4, 6, 13, 15, 16, 17, 18] and the references therein). Another reason for the extensive literature of this type of problems is the similarity of this equation with Navier-Stokes equations since both include nonlinear terms of type: unknown function multiplied by a first derivative and both also contain higher order terms multiplied by a parameter**.**

Since analytical solutions of the Burger's equation were obtained for restricted values of kinematics viscosity [4, 10], it is of interest from the numerical viewpoints to solve this equation for various values of viscosity and make comparison of numerical solution with analytical ones. Therefore, the numerical solution of Burger's equation for more than three decades, has been a very active area of research in mathematics, especially for finite difference and finite element methods. However, in the finite difference regime, the diffusion term can be implemented via a backward Euler or upwind difference expression for the convective first derivative term, which essentially introduces a truncation error which has the same for as the diffusion term. Moreover, in many cases of physical interest, the equation is advection-dominated; that is, $u$ is much larger than ν. Therefore, whether the global structure of the equation is more elliptic or parabolic depends on the relative magnitude of one term against the other, the numerical algorithms implemented for the solution of this problem and the techniques used for their analysis, tend to be very different. Really, the standard numerical techniques do not work well for these types of problems, so that the mathematicians think of other methods to deal with them. One of the new algorithms designed for dealing with such problems is the suggested scheme in this work, which is based on combining the method of characteristics with Roth method (the method of lines) to attack the problem of nonlinearity.

The main contribution of this paper is to use the analysis of [5, 7] to introduce and prove the convergence of a competitive numerical scheme to solve Burger's problem. In addition, we obtain error estimates for the solution of the full discretization scheme. The novelty of our algorithm is in the way we discretized the equation with respect to the time variable by backward Euler formula and then we compensate the convection term by the characteristic method to linearize the system of algebraic equations that result from the space discretization scheme.

The paper is organized as follows: Section 2 contains the general formulation of the problem under investigation. Section 3 is devoted to describe and analyze a time discretization scheme. The convergence of the discrete sequence of iterations is shown in Section 4. Section 5 concerns the error estimates for the approximate solution. Finally, in Section 6, the proposed scheme is directly applicable to solve some numerical examples to support the efficiency of the suggested numerical scheme.

## 2. Notations, assumptions and definitions

In the sequel, we will denote by $(.\,,.)$ either the standard inner product in $L_2 = L_2(\Omega)$ or the pairing between $V \equiv H_0^1(\Omega)$ and $V^* \equiv H^{-1}$ (see e.g. [11, 13]). We use the symbols $|\,.\,|$, $\|\,.\,\|$ and $\|\,.\,\|_*$ as the norms in $L_2(\Omega)$, $V$ , $V^*$, respectively. By $\rightarrow, \xrightarrow{w}$, we mean the strong and weak convergence. Also, we introduce some notations concerning the time discretization of our problem.

$$\delta z_i = \frac{z_i - z_{i-1}}{\tau}, \qquad \overset{-i}{z} = \frac{1}{\tau}\int_{I_i} z(.,t)dt, \qquad 1 \le i \le n,$$

For any given family $\{z_i\}_{i=0}^n$. The following elementary relations will be used in the following analysis:

$$2\sum_{i=1}^{n} a_j \sum_{i=1}^{j} a_i = \left[\sum_{j=1}^{n} a_j\right]^2 + \sum_{j=1}^{n} a_j^2., \tag{2.1}$$

$$\|z\|_{L_4(\Omega)}^4 \le \|z\|^2 |z|^2 |\Omega| \qquad \forall z \in C_0^1(\Omega), \tag{2.2}$$

and Young's inequality

$$ab \le \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2 \qquad \forall a, b \in \square\ , \tag{2.3}$$

where $\varepsilon$ is small constant. We will assume, throughout this work, the following hypotheses on the given data.

**(H1)** The functions $f:\ \Omega \times I \to R$ is Lipschitz continuous in the sense of

$$|f(x,\ t) - f(x,\ t')| \le c\ |t - t'|, \qquad \forall\ t,\ t' \in I \tag{2.4}$$

**(H2)** $u_0 \in H_0^1(\Omega)$

Under these assumptions, we can define the variational solution of problem (1.1)-(1.2)

**Definition 2.1** The measurable function $u \in C(I; L_2(\Omega)) \cap L_2(I;V)$ with $\partial_t u \in L_2(I;V^*)$ and $u(x,0) = u_0(x)$ is said to be a weak (variational) solution of (1.1)–(1.3) if and only if the integral identity

$$(u_t, \varphi) + (uu_x, \varphi) + \nu(u_x, \varphi_x) = (f, \varphi), \tag{2.5}$$

holds for all $\varphi \in V$ and a.e. $t \in I$.

***Remark.*** It is evident that for any functions $\alpha, \beta \in H^1(\Omega)$ and $\gamma \in L_2(\Omega)$

$$|(\alpha\beta_x, \gamma)| \leq \|\alpha\|^{1/2} |\alpha|^{1/2} |\beta_x| |\gamma| \leq \|\alpha\| \|\beta\| \|\gamma\|, \tag{2.6}$$

and

$$|(\alpha\alpha_x, \gamma)| = \frac{1}{2} \left| (\alpha^2, \gamma_x) \right| \leq \frac{1}{2} \|\alpha\| \|\gamma\| |\alpha|, \tag{2.7}$$

**3. The semidiscretization scheme. A prioi estimates**

Our main goal is to approximate (1.1)–(1.3) from a numerical point of view and to prove its convergence. The suggested technique is based on the combination of the characteristics and Roth methods. Using a backward Euler difference scheme for the time derivative and then applying the characteristic method to compensate the convection term which is discretized explicitly so that the underlying equation is converted into a linear system of algebraic equations that easily solved numerically at each subsequent time level. To this purpose, let *n* be a positive integer. Subdivide the time interval *I* by the points $t_i$, where $t_i = i\tau$, $\tau = T/n$, $i = 0, 1, \cdots, n$. The suggested discretization scheme of problem (2.5) consists of the following problem (in the weak sense):

Find $w_i \cong u(., t_i) \in V, \quad i = 1, \cdots, n$ such that

$$w_0 = u_0 \qquad \text{in } \Omega, \tag{3.1}$$

$$w_{i-1}^* = \tilde{w}_{i-1}(x - \tau w_{i-1}) \tag{3.2}$$

$$(w_i - w_{i-1}^*, \varphi) + \tau\nu(\partial_x w_i, \partial_x \varphi) = \tau(f_i, \varphi) \qquad \forall \varphi \in V, \tag{3.3}$$

where $\partial_x$ denotes to the derivative with respect to *x*, $f_i = f(x, t_i)$ and $\tilde{w} \in W^{1,2}(\Omega)$ is an extension of $w \in W^{1,2}(\Omega^*), \quad \Omega^* \supset \bar{\Omega}$, satisfying

$$\| \tilde{w} \|_{H^1(\Omega^*)} \leq C \| w \|_{H^1(\Omega)} \tag{3.4}$$

The existence of a weak solution $w_i \in V$ is guaranteed by Lax–Milgram argument. By means of $w_i$, ($i = 0, 1, \cdots, n$) determined by the proposed scheme (3.1)-(3.3) in each time step $t_i$, we introduce the following piecewise linear functions (Rothe functions)

$$w^n(0) = u_0, \quad w^n(t) = w_{i-1} + (t - t_{i-1})\delta w_i, \qquad \text{for } t \in \left(t_{i-1}, t_i\right], \quad i = 1, 2, \cdots, n, \tag{3.5}$$

and the corresponding step function

$$\bar{w}^n(0) = u_0, \quad \bar{w}^n(t) = w_i \quad \text{for } t \in (t_{i-1}, \ t_i] \tag{3.6}$$

Using the notation of Rothe function and its corresponding step function, a piecewise constant interpolation of equation (3.3) over *I* yields

$$(\partial_t w^n, \varphi) + (\bar{w}_\tau^n \partial_x \bar{w}_\tau^n, \varphi) + \nu(\partial_x \bar{w}^n, \partial_x \varphi) = (\bar{f}^n, \varphi), \quad \forall \varphi \in V \tag{3.7}$$

where $\bar{w}_\tau^n(t) = \bar{w}^n(t - \tau)$, $t_{i-1} \leq t \leq t_i$ and $\bar{f}^n = f(., \bar{t}^n)$ with $\bar{t}^n = t_i$.

In order to show the stability of the discrete solution and prove the convergence results, we shall derive some a priori estimates.

***Lemma 3.1*** Under the assumptions (H1) and (H2), there exists a positive constant $C$ such that

$$\sum_{i=1}^{s}\tau\left|\delta w_i\right|^2+\max_{s}\left\|w_s\right\|^2+\sum_{i=1}^{s}\left\|w_i-w_{i-1}\right\|^2\le C, \tag{3.8}$$

for any $s$.

*Proof.* Let us choose $\varphi=\tau\delta w_i$ in (3.3) and summing over $i$ from 1 to $s$, we obtain

$$\sum_{i=1}^{s}\left(\delta w_i,\tau\delta w_i\right)+\sum_{i=1}^{s}\left(\frac{(w_{i-1}-w_{i-1}^{*})}{\tau},\tau\delta w_i\right)+\nu\sum_{i=1}^{s}(\partial_x w_i,\partial_x(w_i-w_{i-1}))=\sum_{i=1}^{s}\tau(f_i,\delta w_i), \tag{3.9}$$

With the aid of (3.2) and Younge's inequality (2.3), we estimate the second term by

$$\left|\sum_{i=1}^{s}\left(\frac{(w_{i-1}-w_{i-1}^{*})}{\tau},\tau\delta w_i\right)\right|\le\frac{1}{2\varepsilon}\sum_{i=1}^{s}\tau\left\|w_i\right\|^2+\frac{\tau}{2\varepsilon}\sum_{i=1}^{s}\left\|w_i-w_{i-1}\right\|^2+2\varepsilon\sum_{i=1}^{s}\tau\left|\delta w_i\right|^2 \tag{3.10}$$

Taking into consideration (2.1), the elliptic term is estimated by

$$\nu\sum_{i=1}^{s}(\partial_x w_i,\partial_x(w_i-w_{i-1}))=\frac{\nu}{2}\left(\left|\partial_x w_s\right|^2-\left|\partial_x u_0\right|^2+\sum_{i=1}^{s}\left|\partial_x\left(w_i-w_{i-1}\right)\right|^2\right)$$

$$=\frac{\nu}{2}\left(\left\|w_s\right\|^2-\left\|u_0\right\|^2+\sum_{i=1}^{s}\left\|w_i-w_{i-1}\right\|^2\right) \tag{3.11}$$

And the last term is estimated as follows

$$\left|\sum_{i=1}^{s}\tau(f_i,\delta w_i)\right|\le\sum_{i=1}^{j}\tau\left|f_i\right|\left|\delta w_i\right|\le\frac{1}{2\varepsilon}\sum_{i=1}^{j}\tau\left|f_i\right|^2+2\varepsilon\sum_{i=1}^{j}\tau\left|\delta w_i\right|^2$$

$$=\frac{1}{2\varepsilon}\sum_{i=1}^{j}\tau\left|f_i-f(0)\right|^2+2\varepsilon\sum_{i=1}^{j}\tau\left|\delta w_i\right|^2\le C_\varepsilon+\varepsilon\sum_{i=1}^{j}\tau\left|\delta w_i\right|^2 \tag{3.12}$$

Collecting (3.9)-(3.12), using ($H_2$), and choosing $\varepsilon$ sufficiently small, we get

$$\sum_{i=1}^{s}\tau\left|\delta w_i\right|^2+\nu\left\|w_s\right\|^2+\nu\sum_{i=1}^{s}\left\|w_i-w_{i-1}\right\|^2\le c_\varepsilon\sum_{i=1}^{j}\tau\left|f_i\right|^2+\varepsilon\sum_{i=1}^{j}\tau\left|\delta w_i\right|^2+$$

$$+\frac{1}{2\varepsilon}\sum_{i=1}^{s}\tau\left\|w_i\right\|^2+\frac{\tau}{2\varepsilon}\sum_{i=1}^{s}\left\|w_i-w_{i-1}\right\|^2 \tag{3.13}$$

Applying Gronwall's inequality and setting $\nu-\frac{\tau}{2\varepsilon}>0$, we conclude the proof.

***Lemma 3.3*** Uniformly with respect to $n$ one has

$$\left\|\partial_t w^n\right\|_{L_2(I;\,L_2(\Omega))}^2\le C\quad\left\|\overline{w}^n\right\|_{C(I;V)}^2\le C,\quad\left\|w^n\right\|_{C(I;V)}^2\le C \tag{3.14}$$

$$\left\|w^n-\overline{w}^n\right\|_{L_2(I;\,L_2(\Omega))}^2+\left\|w^n-\overline{w}_\tau^n\right\|_{L_2(I;\,L_2(\Omega))}^2\le\frac{C}{n^2}, \tag{3.15}$$

$$\left\| w^n - \overline{w}^n \right\|^2_{L_2(I;V)} + \left\| w^n - \overline{w}^n_\tau \right\|^2_{L_2(I;V)} \le \frac{C}{n} \tag{3.16}$$

*Proof.* The estimates $(3.14)_1$ and $(3.14)_2$ are a consequence of $(3.8)_1$. By the use of the identity

$$\left| w^n - \overline{w}^n \right| \le 2\left| w_i - w_{i-1} \right|, \tag{3.17}$$

the estimates $(3.14)_2$, $(3.14)_3$, (3.15) and (3.16) are a consequence of $(3.8)_2$, $(3.8)_3$ and the definitions of $w^n$ and $\overline{w}^n$ and thus the proof completes.

## 4. Convergence results

This section is devoted to proving the convergence of the proposed scheme and estimating its accuracy. Before we are able to prove convergence we need to prove the compactness of $\overline{w}^n$ in $L_2(I;L_2(\Omega))$ which is a consequence of the following assertion.

***Lemma 4.1*** The estimate

$$\int_0^{T-z} \left| \overline{w}^n(t+z) - \overline{w}^n(t) \right|^2 \le C\left( z + \frac{1}{n} \right), \tag{4.1}$$

holds uniformly for $0 < z < z_0$ and *n*.

*Proof.* [11] We sum up (3.3) for $i = s+1, \cdots, s+k$ considering $\varphi = (w_{s+k} - w_s)\tau$. Then we sum it up for $s = 1, \cdots, n-k$ and obtain the estimate

$$\sum_{s=0}^{n-k} \left| u_{s+k} - u_s \right|^2 \tau \le Ck\tau, \tag{4.2}$$

Hence for $k\tau \le z \le (k+1)\tau$ we conclude the desired estimate.

***Theorem 4.1*** There exists $u \in L_2(I;\ L_2(\Omega)) \cap H^1(I;\ L_2(\Omega))$ such that

$$\left.\begin{array}{lll} w^n \longrightarrow u & & \text{in } L_2(I;L_2(\Omega)) \\ w^n \xrightarrow{w} u, & \overline{w}^n \xrightarrow{w} u & \text{in } L_2(I;V) \\ \partial_t w^n \xrightarrow{w} \partial_t u & & \text{in } L_2(I;L_2(\Omega)) \end{array}\right\}, \tag{4.3}$$

(in the sense of subsequences). Moreover, we have

$$\left\| u - \overline{w}^n \right\|^2_{L_2(I;L_2(\Omega))} + \left\| u - \overline{w}^n \right\|^2_{L_2(I;V)} \le C\tau^2 \tag{4.4}$$

*Proof.* The estimate $(3.14)_2$ implies

$$\int_Q (\overline{w}^n(x+y,\ t)-\overline{w}^n(x,\ t))^2 \le C|y|, \qquad \forall |y| \le y_0$$

Hence, from Lemma 4.1, $\{\overline{w}^n\}$ is compact in $L_2(I;L_2(\Omega))$ because of Kolmogorov's compactness argument. So we can conclude that $w^n \to u$ and $\overline{w}^n \to u$ in $L_2(I;L_2(\Omega))$ and also pointwise in $Q$. Also by the elementary identity (2.2) and the fact that $\overline{w}^n \in L_2(I;\ L_2(\Omega)) \cap C(I;\ V)$, we obtain that $\left(\overline{w}^n\right)^2$ is bounded in $L_2(I;L_2(\Omega))$ and $\left(\overline{w}^n\right)^2$ is weakly convergent in $L_2(I;L_2(\Omega))$. But $\overline{w}^n \longrightarrow u$ in $L_2(I;L_2(\Omega))$, so

$$\left(\overline{w}^n\right)^2 \xrightarrow{w} u^2 \text{ in } L_2(I;L_2(\Omega)) \tag{4.5}$$

Hence, by (3.16), we have

$$\left(\overline{w}_\tau^n\right)^2 \xrightarrow{w} u^2 \text{ in } L_2(I;L_2(\Omega)), \tag{4.6}$$

and this implies that

$$(\overline{w}_\tau^n \partial_x \overline{w}_\tau^n\ , \varphi) \to (uu_x\ , \varphi), \quad \forall \varphi \in V \tag{4.7}$$

It remains to prove that $\partial_t w^n = \partial_t u$. For each $t \in I$, by Lemma 3.3, $\partial_t w^n$ is uniformly bounded in the reflexive Banach space $L_2(I;L_2(\Omega))$ and hence has a subsequence which converges weakly to an element $\xi \in V^*$ (Eberlin-Smulian theorem [12]). Thus

$$\partial_t w^n \overset{w}{\to} \xi \quad \text{in } V^* \quad \forall t \in I \tag{4.8}$$

Using Fubini theorem, we get

$$\left(w^n(t)-u_0, \chi\right) = \int_0^t \left(\partial_t w^n, \chi\right) dt \tag{4.9}$$

Taking the limit as $n \to \infty$, we obtain

$$(u-u_0, \chi) = \int_0^t (\xi, \chi)\, dt, \tag{4.10}$$

which implies

$$\left(u-u_0-\int_0^t \xi(t)dt, \chi\right) = 0 \tag{4.11}$$

Therefore, we have $\partial_t u = \xi$. Using the above discussions and the fact that and hence $\overline{f}^n \to f$ in $L_2(I;\ L_2(\Omega))$ and passing with $n \to \infty$ in (3.7) the proof is complete.

To obtain an error estimate, let us start by introducing the following additional notation $e_u = u(t) - \overline{w}^n$. We now take the difference between (2.5) and (3.7) we easily obtain the equality

$$(\partial_t (u - w^n), \varphi) + (u\partial_x - \overline{w}_\tau^n \partial_x \overline{w}_\tau^n, \varphi) + v(\nabla(u - \overline{w}^n), \nabla\varphi) = ((f - \overline{f}^n), \varphi), \quad \forall\varphi \in V \tag{4.12}$$

Take $\varphi = e_u$ and write the new equation as $I + II + III = IV$. To begin with, we split the parabolic term into two parts $I = I_1 + I_2$, where

$$I_1 = \frac{1}{2}\frac{d}{dt}\left|e_u\right|^2, \tag{4.13}$$

$$\left|I_2\right| \le \left|w^n - \overline{w}^n\right|\left|e_u\right| \le 2\xi\left|w^n - \overline{w}^n\right|^2 + \frac{1}{2\xi}\left|e_u\right|^2 \tag{4.14}$$

The second term of equation (4.12) is bounded by

$$\left|II\right| \le \left|(u\partial_x (u - \overline{w}_\tau^n), e_u)\right| + \left|((u - \overline{w}_\tau^n)\partial_x \overline{w}_\tau^n, e_u)\right| \tag{4.15}$$

Using the fact that $u \in C(I; L_2(\Omega)) \cap L_2(I; V)$, (2.6), (2.7) and Young's inequality the two parts of this inequality yield

$$\left|II\right| \le 2\varepsilon\left\|u - \overline{w}_\tau^n\right\|^2 + \frac{1}{2\varepsilon}\left|e_u\right|^2, \tag{4.16}$$

In view of (H1) and Younge's inequality we can estimate the right hand by

$$\left|IV\right| \le \frac{\kappa}{2}\left|f - \overline{f}^n\right|^2 + \frac{1}{2\kappa}\left|e_u\right|^2 \le \frac{\kappa}{2}\tau^2 + \frac{1}{2\kappa}\left|e_u\right|^2 \tag{4.17}$$

Collecting all the previous bounds, choosing $\varepsilon$ and $\kappa$ sufficiently small and applying the nonlinear Gronwall lemma we obtain (4.4).

**5. Numerical experiment and discussion**

In this section, we shall solve Burger's equation (1.1) in $Q_T \equiv (0,1) \times (0,T)$. We employ an explicit central difference scheme for the space derivative so that we get a full discretization scheme with an error estimation $O(h^2) + O(\tau)$. The boundary and initial conditions we have used in this experiment are $u(0,t) = u(1,t) = 0,\ t > 0$ and $u(x,0) = \sin x,\ 0 \le x \le 1$. We use a spatial finite difference discretization to solve the linear elliptic problem (3.3) with $f = 0$. We shall compare the results obtained by the suggested approximation scheme (3.1)-(3.3) with the exact solution and with other methods which are introduced by [8]. Tables 1-4 display, respectively, the results for $\nu = 10, 1$ and 0.1. It is observed that all the results of the proposed approximation scheme are in good agreement with the exact ones and exhibit the expected convergence. In addition to the simplicity due to the solution of linear systems of equations, in view of these tables, we notice that the convergence rate of our scheme is slightly better than that of [9].

**Table 1.** Comparison between exact and numerical solutions at $t = 0.02$, $\nu = 10$, $\tau = 0.0001$ and $t = 0.01$, $\nu = 10$, $\tau = 0.0001$, respectively.

| $x$ | $t = 0.02,\ \nu = 10,\ \tau = 0.0001$ | | | $t = 0.01,\ \nu = 10,\ \tau = 0.0001$ | | |
|---|---|---|---|---|---|---|
| | Exact solution | Suggested scheme | Hon and Mao[9] | Exact solution | Suggested scheme | Hon and Mao |

| | | | | | | |
|---|---|---|---|---|---|---|
| 0.0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0.1 | 0.0428 | 0.0428 | 0.0433 | 0.1146 | 0.1145 | 0.1152 |
| 0.2 | 0.0815 | 0.0815 | 0.0823 | 0.2182 | 0.2180 | 0.2192 |
| 0.3 | 0.1122 | 0.1124 | 0.1133 | 0.3006 | 0.3008 | 0.3021 |
| 0.4 | 0.1320 | 0.1325 | 0.1333 | 0.3539 | 0.3546 | 0.3556 |
| 0.5 | 0.1389 | 0.1398 | 0.1403 | 0.3727 | 0.3739 | 0.3745 |
| 0.6 | 0.1322 | 0.1334 | 0.1335 | 0.3550 | 0.3566 | 0.3567 |
| 0.7 | 0.1125 | 0.1138 | 0.1136 | 0.3024 | 0.3042 | 0.3039 |
| 0.8 | 0.0818 | 0.0829 | 0.0826 | 0.2200 | 0.2215 | 0.2211 |
| 0.9 | 0.0430 | 0.0436 | 0.0434 | 0.1157 | 0.1166 | 0.1163 |
| 1.0 | 0 | 0 | 0 | 0 | 0 | 0 |

**Table 2.** Comparison between exact and numerical solutions at $t = 0.1$, $\nu = 1$, $\tau = 0.00001$ and $t = 0.5$, $\nu = 0.1$, $\tau = 0.01$, respectively.

| *x* | $t = 0.1,\ \nu = 1,\ \tau = 0.00001$ | | | $t = 0.5,\ \nu = 0.1,\ \tau = 0.01$ | | |
|---|---|---|---|---|---|---|
| | Exact solution | Suggested scheme | Kutluay et al. N=10 [8] | Exact solution | Suggested scheme | Hon and Mao |
| 0.0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0.1 | 0.10954 | 0.1047 | 0.11048 | 0.1099 | 0.0637 | 0.1104 |
| 0.2 | 0.20979 | 0.2019 | 0.21159 | 0.2180 | 0.1302 | 0.2186 |
| 0.3 | 0.29190 | 0.2841 | 0.29435 | 0.3222 | 0.2020 | 0.3227 |
| 0.4 | 0.34792 | 0.3433 | 0.35080 | 0.4190 | 0.2815 | 0.4194 |
| 0.5 | 0.37158 | 0.3719 | 0.37458 | 0.5028 | 0.3686 | 0.5023 |
| 0.6 | 0.35905 | 0.3636 | 0.36189 | 0.5623 | 0.4538 | 0.5618 |
| 0.7 | 0.30991 | 0.3180 | 0.31231 | 0.5759 | 0.4959 | 0.5744 |
| 0.8 | 0.22782 | 0.2365 | 0.22955 | 0.5055 | 0.4479 | 0.5030 |
| 0.9 | 0.12069 | 0.1262 | 0.12160 | 0.3093 | 0.2787 | 0.3059 |
| 1.0 | 0 | 0 | 0 | 0 | 0 | 0 |

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