Mixture Weibull Exponential Distribution for Fitting Failure Times Data

Ibtesam Alsaggaf,[a] Hanan Alzahrani [b]

[a] Department of Statistics, King Abdulaziz University, KSA
ialsaggaf@kau.edu.sa
[b] Science College, King Abdulaziz University, KSA
hmkalzahrani@kau.edu.sa

Abstract. A new mixture model called Weibull exponential mixture model is introduced in this paper. The new model turns out to be quite flexible for analyzing positive data. The maximum likelihood estimates (MLE's) of the parameters of the new mixture model are obtained based on full samples, Type-I and Type-II censored samples. Certain statistical characteristics associated with this distribution are obtained. A simulation study is employed to check the consistency of maximum likelihood estimates. This new distribution may provide better fitting to describe positive data in various scientific fields such as the physical and biological sciences, medicine, meteorology, and engineering.

.

Keywords: Weibull exponential distribution; maximum likelihood estimation, Bias, mean squared error; Type-I censored samples, Type-II censored samples.

1 Introduction

Lifetime data are present widely in many different applications. However, the data in many applications such as economics, engineering, biological studies, environmental sciences, medical sciences, and finance can be considered as data coming from mixture population of two or more different distributions. Much research focus on extending and modifying the existing classical distribution to obtain greater flexibility and adaptability in modeling data from different mixture populations. Recently, the mixture modeling has been employed for modeling data from different mixture populations. This paper shows the idea of generating new model by applying the mixture modeling with Weibull exponential distribution (MWE).

The using of finite mixture models is very old in the history of statistics. It was useful in modeling population heterogeneity, classification, clustering and generalize distribution assumptions. The first use of finite mixture models was in the nineteenth century in a paper by Newcomb (1886) who used it in the context of modeling outliers. Pearson (1894) studied of a mixture of two univariate Gaussian distribution and estimated the parameters of the model using the method of moments. He used the mixture approach to analyze a data set containing ratios of forehead to body lengths for 1,000 crabs. Figueiredo and Jain (2002) used the finite mixture to unsupervised learning models. In (2011), Franco et al. studied the classification of the aging properties of generalized mixtures of two or three Weibull distributions in terms of the mixing weights, scale parameters and a common shape parameter. Razali and Al-Wakeel (2013) used the mixture of two and three Weibull distributions to analyze the data of failure times. Zhang et al. (2014) proposed mixture Weibull proportional hazards model to predict the failure of a mechanical system with multiple failure modes. Elshahat and Mahmoud (2016) studied the mixture of exponentiated-Weibull distribution and estimated the parameters using maximum likelihood estimation. Qutb et al. (2016) obtained the estimation of the parameters, reliability and hazard rate functions of the mixture of two Weibull distributions with a common shape parameter, based on the generalized order statistics. Huang et al. (2017) discussed likelihood method for finite multivariate Gaussian mixture models. Zong et al. (2018) studied a deep autoencoding Gaussian mixture model for unsupervised anomaly. McLachlan et al. (2019) provided the methodological and theory for the applications of finite mixture models and discussed the role of mixture models in clustering of independent and identically distributed data. They also used the maximum likelihood estimation and the moment estimation methods for parametric mixture models. Teamah et al. (2020) introduced a new mixture distribution as a result of mixing Fr´echet-Weibull distribution with exponential distribution; it is called Fr´echet-Weibull mixture exponential distribution and used the maximum likelihood estimation for estimating the parameters of the mixture distribution.

In this paper we will form and study a mixture of two component of Weibull exponential distribution. Also, we provide a comprehensive comparison of different estimation methods for the model parameters. The following criteria are used for comparison: the Bias and the mean squared error (MSE). Simulated data are used to study the performance of model estimators.

2 Two-Components Mixture Weibull Exponential Distribution

The probability density function (PDF) of the Weibull exponential distribution is given as

$$f(x) = \frac{1}{\beta^c}\, c\, \lambda^c\, x^{c-1}\, e^{-\left(\frac{\lambda x}{\beta}\right)^c}, x \geq 0, c, \lambda, \beta > 0, \quad (1)$$

and the cumulative distribution function (CDF) is given by

$$F(x) = 1 - e^{-\left(\frac{\lambda x}{\beta}\right)^c} \quad (2)$$

The hazard rate function is given as

$$h(x) = \frac{1}{\beta^c}\, c\, \lambda^c\, x^{c-1}. \quad (3)$$

A density function for the mixture of two components densities with mixing proportions p is defined as

$$f(x) = p f_1(x) + (1-p)\, f_2(x),$$

$$= p\left[\frac{1}{{\beta_1}^{c_1}}\, c_1\, {\lambda_1}^{c_1} x^{c_1-1} e^{-\left(\frac{\lambda_1 x}{\beta_1}\right)^{c_1}}\right] + (1-p)\left[\frac{1}{{\beta_2}^{c_2}}\, c_2\, {\lambda_2}^{c_2}\, x^{c_2-1} e^{-\left(\frac{\lambda_2 x}{\beta_2}\right)^{c_2}}\right], \quad (4)$$

where p satisfies the condition, $0 \leq p \leq 1$. The CDF for the mixture model is defined as

$$F(x) = pF_1(x) + (1-p)F_2(x),$$

$$= p\left[1 - e^{-\left(\frac{\lambda_1 x}{\beta_1}\right)^{c_1}}\right] + (1-p)\left[1 - e^{-\left(\frac{\lambda_2 x}{\beta_2}\right)^{c_2}}\right]. \quad (5)$$

The reliability function for the mixture model is given as

$$R(x) = pe^{-\left(\frac{\lambda_1 x}{\beta_1}\right)^{c_1}} + (1-p)e^{-\left(\frac{\lambda_2 x}{\beta_2}\right)^{c_2}}. \quad (6)$$

The hazard rate function is given as

$$h(x) = p\, c_1 \left(\frac{\lambda_1}{\beta_1}\right)^{c_1} x^{c_1-1} + (1-p)c_2 \left(\frac{\lambda_2}{\beta_2}\right)^{c_2} x^{c_2-1}. \quad (7)$$

The reversed hazard rate function is given as

$$rh(x) = p\frac{c_1\left(\frac{\lambda_1}{\beta_1}\right)^{c_1} x^{c_1-1}\, e^{-\left(\frac{\lambda_1 x}{\beta_1}\right)^{c_1}} +}{1 - e^{-\left(\frac{\lambda_1 x}{\beta_1}\right)^{c_1}}} + (1-p)\frac{c_2\left(\frac{\lambda_2}{\beta_2}\right)^{c_2} x^{c_2-1} e^{-\left(\frac{\lambda_2 x}{\beta_2}\right)^{c_2}}}{1 - e^{-\left(\frac{\lambda_2 x}{\beta_2}\right)^{c_2}}} \quad (8)$$

The quantile function is given as

$$Q(u) = p\left[\frac{\beta_1}{\lambda_1}(-Log(1-u))^{1/c_1}\right] + (1-p)\left[\frac{\beta_1}{\lambda_1}(-Log(1-u))^{1/c_1}\right].$$

Graphical description

The plots of PDF and the hazard function (HZ) are displayed for different values of parameters in Figure1. The figure shows several forms for PDF and HZ curve. This indicates this new distribution is flexible and may be suitable for a different type of data.

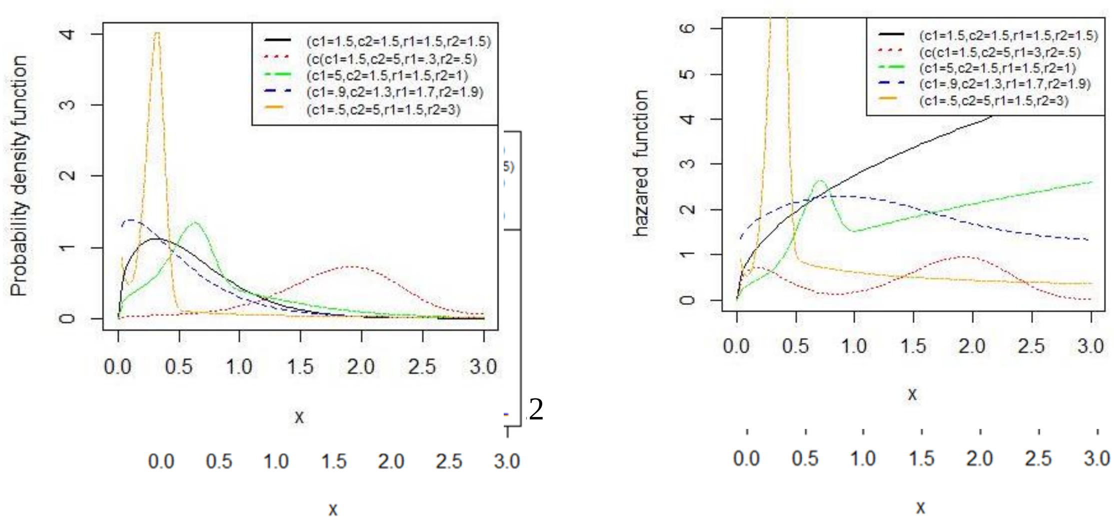

Figure 1 Plots of PDF and HZ functions of the MWE distribution.

3 General Properties
3.1Moment

The finite mixture of the r^{th} moments of the two components is represented as

$$\acute{\mu}_r = \sum_{j=1}^{2} p_j \, \mu_{il} = \int p \, c_1 \, {\lambda_1}^{c_1} \, {\beta_1}^{-c_1} \, x^{r+c_1-1} \, e^{-\left(\frac{\lambda_1 x}{\beta_1}\right)^{c_1}} dx$$

$$+ \int (1-p) \quad c_2 \, {\lambda_2}^{c_2} \, {\beta_2}^{-c_2} \, x^{r+c_2-1} \, e^{-\left(\frac{\lambda_2 x}{\beta_2}\right)^{c_2}} dx. \qquad (9)$$

$$E(x^r) = p \left(\frac{\beta_1}{\lambda_1}\right)^r \sqrt{\left(\frac{r}{c_1}+1\right)} + (1-p) \left(\frac{\beta_2}{\lambda_2}\right)^r \sqrt{\left(\frac{r}{c_2}+1\right).} \qquad (10)$$

The mean is given when $r = 1$ as follows

$$\acute{\mu}_1 = p \, \frac{\beta_1}{\lambda_1} \sqrt{\left(\frac{1}{c_1}+1\right)} + (1-p) \, \frac{\beta_2}{\lambda_2} \sqrt{\left(\frac{1}{c_2}+1\right).} \qquad (11)$$

The variance is given as follows

$$\sigma^2 = E(x^2) - {\acute{\mu}_1}^2.$$

The moments generating function of MWE distribution is expressed as

$$M_X(t) = \int e^{tx} f_j(x) dx$$

$$= \int e^{tx}\left\{p\left[\frac{1}{{\beta_1}^{c_1}}\ c_1\ {\lambda_1}^{c_1}x^{c_1-1}e^{-\left(\frac{\lambda_1\, x}{\beta_1}\right)^{c_1}}\right] + (1 - p)\ \left[\frac{1}{{\beta_2}^{c_2}}\ c_2\ {\lambda_2}^{c_2}\, x^{c_2-1}e^{-\left(\frac{\lambda_2 x}{\beta_2}\right)^{c_2}}\right]\right\} dx.$$

$M_X(t) = p[M_{X1}(t)] + (1-p)[M_{X2}(t)]$.

So, the moments generating function of MWE distribution can be written as

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\left\{p\left(\frac{\beta_1}{\lambda_1}\right)^r \sqrt{\left(\frac{r}{c_1}+1\right)}\ + (1-p)\left(\frac{\beta_2}{\lambda_2}\right)^r \sqrt{\left(\frac{r}{c_2}+1\right)}\right\}. \quad (12)$$

3.2 Incomplete Moments

The rth incomplete moment of MWE distribution is given as

$$T_r(z) = \sum_{j=1}^{2} p_j \mu_{jr} = \int^{z} x^r\ p\, c_1\ {\lambda_1}^{c_1}\, {\beta_1}^{-c_1}\ x^{c_1-1}\ e^{-\left(\frac{\lambda_1 x}{\beta_1}\right)^{c_1}}\ dx + \int^{z} x^r (1-p)\ \ c_2\ {\lambda_2}^{c_2}\ {\beta_2}^{-c_2}\ x^{c_2-1}\ e^{-\left(\frac{\lambda_2 x}{\beta_2}\right)^{c_2}}\ dx. \quad (13)$$

The first incomplete moment of a finite mixture of k components equal

$$T_1(z) = \sum_{j=1}^{2} p_j \mu_{j1} = \int^{z} x\ p\ c_1\ {\lambda_1}^{c_1}\ {\beta_1}^{-c_1}\ x^{c_1-1}\ e^{-\left(\frac{\lambda_1 x}{\beta_1}\right)^{c_1}}\ dx + \int^{z} x\ (1-p)\ \ c_2\ \ {\lambda_2}^{c_2}\ {\beta_2}^{-c_2}\ x^{c_2-1}\ e^{-\left(\frac{\lambda_2 x}{\beta_2}\right)^{c_2}}\ dx. \quad (14)$$

3.3 Mean Deviations

The mean deviation about the mean μ of MWR distribution is given as

$$\delta 1 = \int_D\ |x-\mu|\left[p\left(c_1\ {\lambda_1}^{c_1}\, {\beta_1}^{-c_1}\ x^{c_1-1}\ e^{-\left(\frac{\lambda_1 x}{\beta_1}\right)^{c_1}}\right) + (1-p)\left(c_2\ {\lambda_2}^{c_2}\ {\beta_2}^{-c_2}\ x^{c_2-1} e^{-\left(\frac{\lambda_2 x}{\beta_2}\right)^{c_2}}\right)\right] dx.$$

$$\delta 1 = p\left(c_1\ {\lambda_1}^{c_1}\, {\beta_1}^{-c_1}\ x^{c_1-1}\ e^{-\left(\frac{\lambda_1 x}{\beta_1}\right)^{c_1}}\right) + (1-p)\left(c_2\ {\lambda_2}^{c_2}\ {\beta_2}^{-c_2}\ x^{c_2-1} e^{-\left(\frac{\lambda_2 x}{\beta_2}\right)^{c_2}}\right), \quad (15)$$

and the mean deviation about the median M equals

$$\delta 2 = \int_D |x - M| \left[p \left(c_1\, \lambda_1{}^{c_1}\, \beta_1{}^{-c_1}\, x^{c_1 - 1}\, e^{-\left(\frac{\lambda_1 x}{\beta_1}\right)^{c_1}} \right) + (1-p) \left(c_2\, \lambda_2{}^{c_2}\, \beta_2{}^{-c_2}\, x^{c_2 - 1}\, e^{-\left(\frac{\lambda_2 x}{\beta_2}\right)^{c_2}} \right) \right] dx. \quad (16)$$

Since, the median is given as $F(x; \lambda, c, \beta) = \frac{1}{2}$, these forms can be written as

$$\delta 1 = 2\mu\, F(x; \lambda, c, \beta) - 2T_1(\mu),$$

and

$$\delta 2 = \mu - 2T_1(M),$$

where $T_1(z)$ is the first incomplete moment of X obtained from (14). Therefore

$$\delta 1_j = 2\mu_j F_j(x; \lambda_j, c_j, \beta_j) - 2T_1(\mu_j),$$

and

$$\delta 2_j = \mu_j - 2T_1(M_j),$$

where

$$\delta 1 = 2\mu \left(p \left[1 - e^{-\left(\frac{\lambda_1 x}{\beta_1}\right)^{c_1}} \right] + (1-p) \left[1 - e^{-\left(\frac{\lambda_2 x}{\beta_2}\right)^{c_2}} \right] \right) - 2T_1(\mu),$$

and

$$\delta 2 = \mu - 2T_1(M).$$

3.4 Rényi Entropy

The Rényi entropy of MWE model is expressed as

$$H_R^S(x) = \frac{1}{1-S} logp \left(\int_D \left[\left(p\, c_1\, \lambda_1{}^{c_1}\, \beta_1{}^{-c_1}\, x^{c_1 - 1}\, e^{-\left(\frac{\lambda_1 x}{\beta_1}\right)^{c_1}} \right) + (1-p) \left(c_2\, \lambda_2{}^{c_2}\, \beta_2{}^{-c_2}\, x^{c_2 - 1}\, e^{-\left(\frac{\lambda_2 x}{\beta_2}\right)^{c_2}} \right) \right]^S dx \right). \quad (17)$$

It is a difficult problem to obtain $H_R^S(x)$ in closed-form for the mixture model.

3.5 Shannon entropy

The Shannon entropy of X is represented as

$$H_s(x) = E_x\{-log(f(x; \lambda, c, \beta))\},$$

where the log-likelihood function is given as

$$log\left[f_j\left(x,c_j,\lambda_j,\beta_j\right)\right]=log\,p+log\;\;c_1+c_1 log\;\lambda_1-c_1 log\;\beta_1$$

$$+(c_1-1)\,log\;\;x\;-\left(\frac{\lambda_1 x}{\beta_1}\right)^{c_1}+log\;(1-p)+log\;c_2$$

$$+c_2\;log\;\lambda_2-c_2 log\;\beta_2+(c_2-1)\;log\;x\;-\left(\frac{\lambda_2 x}{\beta_2}\right)^{c_2}.$$

Thus, it can be reduced to

$$H_s(x)=\;-\,log\,p-log\;\;c_1-c_1 log\;\lambda_1+c_1 log\;\beta_1$$

$$-(c_1-1)E(log\,X)+E\left(\frac{\lambda_1 X}{\beta_1}\right)^{c_1}-log\;(1-p)-log\;c_2$$

$$-c_2\;log\;\lambda_2+c_2 log\;\beta_2-(c_2-1)E(log\,X)-E\left(\frac{\lambda_2 X}{\beta_2}\right)^{c_2}. \qquad (18)$$

3.6 Distribution of order statistic

The r^{th} order statistics for the MWR distribution can be written as

$$f_{r,n}(x)=\frac{n!}{(n-r)!\,(r-1)!}\sum_{i=0}^{n-r}(-1)^i\binom{n-r}{i}$$

$$\times\left(p\;c_1\;{\lambda_1}^{c_1}\,{\beta_1}^{-c_1}\;x^{c_1-1}\;e^{-\left(\frac{\lambda_1 x}{\beta_1}\right)^{c_1}}+(1-p)\;c_2\;{\lambda_2}^{c_2}\;{\beta_2}^{-c_2}\;x^{c_2-1}e^{-\left(\frac{\lambda_2 x}{\beta_2}\right)^{c_2}}\right)$$

$$\times\left[p\left(1-\;e^{-\left(\frac{\lambda_1 x}{\beta_1}\right)^{c_1}}\right)+(1-p)\left(1-e^{-\left(\frac{\lambda_2 x}{\beta_2}\right)^{c_2}}\right)\right]^{r+i-1}. \qquad (19)$$

Special Cases:

If $r\;=1$, in (19), the PDF of the smallest order statistic can be obtained.

If $r=\;n$, in (19), the PDF of the largest order statistic can be obtained.

If $r\;=\;\frac{n+1}{2}$, in (19), the PDF of the median observable in the odd sample size case can be obtained.

4 Estimation of Mixture Weibull Exponential Distribution

In this section, the parameters of MWE distribution is estimated by maximum likelihood estimation method with complete sample and censoring samples of Type I and Type II.

4.1 Maximum likelihood estimation based on complete sample

If $x_1, x_2, \ldots, x_n$ is a random sample of size n from the MWE distribution, then the log likelihood function for the vector of parameters $\theta_j = (c_j, \lambda_j, \beta_j)$ is given as

$$\ell(c_j, \lambda_j, \beta_j) = \sum_{i=1}^{n}\left[log\left(\sum_{j=1}^{k} p_j\, c_j \left(\frac{\lambda_j}{\beta_j}\right)^{c_j} x_i^{c_j-1}\, e^{-\left(\frac{\lambda_j x_i}{\beta_j}\right)^{c_j}}\right)\right]. \qquad (20)$$

The MLEs can be computed by differentiating (20) with respect to each parameter as follows

$$\frac{\partial \ell}{\partial c_j} = \sum_{i=1}^{n}\left[\frac{p_j\left(\frac{\lambda_j}{\beta_j}\right)^{c_j} x_i^{c_j-1}\, e^{-\left(\frac{\lambda_j x_i}{\beta_j}\right)^{c_j}}\left[1+c_j ln\left(\frac{\lambda_j x_i}{\beta_j}\right)-c_j\left(\frac{\lambda_j x_i}{\beta_j}\right)^{c_j} ln\left(\frac{\lambda_j x_i}{\beta_j}\right)\right]}{\sum_{j=1}^{k} p_j\, c_j\left(\frac{\lambda_j}{\beta_j}\right)^{c_j} x_i^{c_j-1}\, e^{-\left(\frac{\lambda_j x_i}{\beta_j}\right)^{c_j}}}\right], (21)$$

$$\frac{\partial \ell}{\partial \lambda_j} = \sum_{i=1}^{n}\left[\frac{p_j c_j(\lambda_j x_i)^{c_j-1}\left(\frac{1}{\beta_j}\right)^{c_j} e^{-\left(\frac{\lambda_j x_i}{\beta_j}\right)^{c_j}}\left[1-\left(\frac{\lambda_j x_i}{\beta_j}\right)^{c_j}\right]}{\sum_{j=1}^{k} p_j\, c_j\left(\frac{\lambda_j}{\beta_j}\right)^{c_j} x_i^{c_j-1}\, e^{-\left(\frac{\lambda_j x_i}{\beta_j}\right)^{c_j}}}\right]' \qquad (22)$$

$$\frac{\partial \ell}{\partial \beta_j} = -\sum_{i=1}^{n}\left[\frac{p_j\frac{c_j^2}{\beta_j}\left(\frac{\lambda_j}{\beta_j}\right)^{c_j} x_i^{c_j-1}\, e^{-\left(\frac{\lambda_j x_i}{\beta_j}\right)^{c_j}}\left[1-\left(\frac{\lambda_j x_i}{\beta_j}\right)^{c_j}\right]}{\sum_{j=1}^{k} p_j\, c_j\left(\frac{\lambda_j}{\beta_j}\right)^{c_j} x_i^{c_j-1}\, e^{-\left(\frac{\lambda_j x_i}{\beta_j}\right)^{c_j}}}\right]. \; (23)$$

Where $j = 1,2$, and $p_1 = p, p_2 = 1 - p$. The MLEs for each parameter can be derived either by solving the system of non-linear equations (21), (22) and (23) numerically or by maximizing (20) by optimization techniques sing the programming language R

4.2 Maximum likelihood estimation based on Type I censored samples

If $x_1, x_2, \ldots, x_n$ is a random sample of size n from the MWE distribution, then the log likelihood function of Type I censored sample for the vector of parameters $\theta_j = (c_j, \lambda_j, \beta_j)$ is given as

$$\ell(c_j,\lambda_j,\beta_j) = \sum_{i=1}^{m}\left[\delta_i \log\left(\sum_{j=1}^{k} p_j\, c_j \left(\frac{\lambda_j}{\beta_j}\right)^{c_j} x_i^{\,c_j-1}\, e^{-\left(\frac{\lambda_j x_i}{\beta_j}\right)^{c_j}}\right) + (1-\delta_i)\log\left(\sum_{j=1}^{k} p_j\, e^{-\left(\frac{\lambda_j x_m}{\beta_j}\right)^{c_j}}\right)\right]. \quad (24)$$

The MLEs can be computed by differentiating (24) with respect to each parameter as follows

$$\frac{\partial \ell}{\partial c_j} = \sum_{i=1}^{m}\left[\frac{\delta_i\, p_j \left(\frac{\lambda_j}{\beta_j}\right)^{c_j} x_i^{\,c_j-1}\, e^{-\left(\frac{\lambda_j x_i}{\beta_j}\right)^{c_j}} \left[1 + c_j \ln\left(\frac{\lambda_j x_i}{\beta_j}\right) - c_j\left(\frac{\lambda_j x_i}{\beta_j}\right)^{c_j} \ln\left(\frac{\lambda_j x_i}{\beta_j}\right)\right]}{\sum_{j=1}^{k} p_j\, c_j \left(\frac{\lambda_j}{\beta_j}\right)^{c_j} x_i^{\,c_j-1}\, e^{-\left(\frac{\lambda_j x_i}{\beta_j}\right)^{c_j}}} - \frac{(1-\delta_i)\, p_j\, e^{-\left(\frac{\lambda_j x_m}{\beta_j}\right)^{c_j}} \left(\frac{\lambda_j x_m}{\beta_j}\right)^{c_j} \ln\left(\frac{\lambda_j x_m}{\beta_j}\right)}{\sum_{j=1}^{k} p_j\, e^{-\left(\frac{\lambda_j x_m}{\beta_j}\right)^{c_j}}}\right], \quad (25)$$

$$\frac{\partial \ell}{\partial \lambda_j} = \sum_{i=1}^{m}\left[\frac{\delta_i\, p_j c_j (\lambda_j x_i)^{c_j-1} \left(\frac{1}{\beta_j}\right)^{c_j} e^{-\left(\frac{\lambda_j x_i}{\beta_j}\right)^{c_j}} \left[1 - \left(\frac{\lambda_j x_i}{\beta_j}\right)^{c_j}\right]}{\sum_{j=1}^{k} p_j\, c_j \left(\frac{\lambda_j}{\beta_j}\right)^{c_j} x_i^{\,c_j-1}\, e^{-\left(\frac{\lambda_j x_i}{\beta_j}\right)^{c_j}}} - \frac{(1-\delta_i)\, p_j c_j\, \lambda_j^{\,c_j-1} \left(\frac{x_m}{\beta_j}\right)^{c_j} e^{-\left(\frac{\lambda_j x_m}{\beta_j}\right)^{c_j}}}{\sum_{j=1}^{k} p_j\, e^{-\left(\frac{\lambda_j x_m}{\beta_j}\right)^{c_j}}}\right], \quad (26)$$

$$\frac{\partial \ell}{\partial \beta_j} = \sum_{i=1}^{m}\left[\frac{-\delta_i\, p_j \frac{c_j^{\,2}}{\beta_j} \left(\frac{\lambda_j}{\beta_j}\right)^{c_j} x_i^{\,c_j-1}\, e^{-\left(\frac{\lambda_j x_i}{\beta_j}\right)^{c_j}} \left[1 - \left(\frac{\lambda_j x_i}{\beta_j}\right)^{c_j}\right]}{\sum_{j=1}^{k} p_j\, c_j \left(\frac{\lambda_j}{\beta_j}\right)^{c_j} x_i^{\,c_j-1}\, e^{-\left(\frac{\lambda_j x_i}{\beta_j}\right)^{c_j}}} + \frac{(1-\delta_i)\, p_j \frac{c_j}{\beta_j} \left(\frac{\lambda_j x_m}{\beta_j}\right)^{c_j} e^{-\left(\frac{\lambda_j x_m}{\beta_j}\right)^{c_j}}}{\sum_{j=1}^{k} p_j\, e^{-\left(\frac{\lambda_j x_m}{\beta_j}\right)^{c_j}}}\right]. \quad (27)$$

Where $j = 1,2$, and $p_1 = p, p_2 = 1 - p$. The MLEs for each parameter can be derived either by solving the system of non-linear equations (25), (26) and (27) numerically or by maximizing (24) by optimization techniques using the programming language R

4.3 Maximum likelihood estimation based on Type II censored samples

If $x_1, x_2, \ldots, x_n$ is a random sample of size n from the MWE distribution, then the log likelihood function of Type II censored sample for the vector of parameters $\theta_j = (c_j, \lambda_j, \beta_j)$ is given as

$$\ell(c_j, \lambda_j, \beta_j) = log \frac{n!}{(n-m)!} + \sum_{i=1}^{m} log\left(\sum_{j=1}^{k} p_j\, c_j \left(\frac{\lambda_j}{\beta_j}\right)^{c_j} x_i^{c_j-1}\, e^{-\left(\frac{\lambda_j x_i}{\beta_j}\right)^{c_j}}\right)$$

$$+(n-m)\, log\left(\sum_{j=1}^{k} p_j\, e^{-\left(\frac{\lambda_j x_m}{\beta_j}\right)^{c_j}}\right). \quad (28)$$

The MLEs can be computed by differentiating (28) with respect to each parameter as follows

$$\frac{\partial \ell}{\partial c_j} = \sum_{i=1}^{m} \frac{p_j \left(\frac{\lambda_j}{\beta_j}\right)^{c_j} x_i^{c_j-1}\, e^{-\left(\frac{\lambda_j x_i}{\beta_j}\right)^{c_j}} \left[1 + c_j ln\left(\frac{\lambda_j x_i}{\beta_j}\right) - c_j \left(\frac{\lambda_j x_i}{\beta_j}\right)^{c_j} ln\left(\frac{\lambda_j x_i}{\beta_j}\right)\right]}{\sum_{j=1}^{k} p_j\, c_j \left(\frac{\lambda_j}{\beta_j}\right)^{c_j} x_i^{c_j-1}\, e^{-\left(\frac{\lambda_j x_i}{\beta_j}\right)^{c_j}}}$$

$$- \frac{(n-m)\, p_j\, e^{-\left(\frac{\lambda_j x_m}{\beta_j}\right)^{c_j}} \left(\frac{\lambda_j x_m}{\beta_j}\right)^{c_j} ln\left(\frac{\lambda_j x_m}{\beta_j}\right)}{\sum_{j=1}^{k} p_j\, e^{-\left(\frac{\lambda_j x_m}{\beta_j}\right)^{c_j}}}, \quad (29)$$

$$\frac{\partial \ell}{\partial \lambda_j} = \sum_{i=1}^{m} \frac{p_j c_j (\lambda_j x_i)^{c_j-1} \left(\frac{1}{\beta_j}\right)^{c_j} e^{-\left(\frac{\lambda_j x_i}{\beta_j}\right)^{c_j}} \left[1 - \left(\frac{\lambda_j x_i}{\beta_j}\right)^{c_j}\right]}{\sum_{j=1}^{k} p_j\, c_j \left(\frac{\lambda_j}{\beta_j}\right)^{c_j} x_i^{c_j-1}\, e^{-\left(\frac{\lambda_j x_i}{\beta_j}\right)^{c_j}}}$$

$$- \frac{(n-m)\, p_j c_j\, \lambda_j^{\,c_j-1} \left(\frac{x_m}{\beta_j}\right)^{c_j} e^{-\left(\frac{\lambda_j x_m}{\beta_j}\right)^{c_j}}}{\sum_{j=1}^{k} p_j\, e^{-\left(\frac{\lambda_j x_m}{\beta_j}\right)^{c_j}}}, \quad (30)$$

$$\frac{\partial \ell}{\partial \beta_j} = -\sum_{i=1}^{m} \frac{p_j \frac{{c_j}^2}{\beta_j} \left(\frac{\lambda_j}{\beta_j}\right)^{c_j} x_i^{\;c_j-1} e^{-\left(\frac{\lambda_j x_i}{\beta_j}\right)^{c_j}} \left[1 - \left(\frac{\lambda_j x_i}{\beta_j}\right)^{c_j}\right]}{\sum_{j=1}^{k} p_j\, c_j \left(\frac{\lambda_j}{\beta_j}\right)^{c_j} {x_i}^{c_j-1}\, e^{-\left(\frac{\lambda_j x_i}{\beta_j}\right)^{c_j}}} + \frac{(n-m)\, p_j \frac{c_j}{\beta_j} \left(\frac{\lambda_j x_m}{\beta_j}\right)^{c_j} e^{-\left(\frac{\lambda_j x_m}{\beta_j}\right)^{c_j}}}{\sum_{j=1}^{k} p_j\, e^{-\left(\frac{\lambda_j x_m}{\beta_j}\right)^{c_j}}}. \quad (31)$$

Where $j = 1{,}2$, and $p_1 = p, p_2 = 1 - p$. The MLEs for each parameter can be derived either by solving the system of non-linear equations (29), (30) and (31) numerically or by maximizing (28) by optimization techniques using the programming language R.

4.4 Estimation of the reliability, hazard, and reversed hazard rate functions

The invariance property of the ML estimators enables us to obtain the ML estimators of the reliability, hazard rate and reversed hazard rate functions by replacing the parameters c_j, λ_j and β_j by their ML estimators in (21), (22) and (23) or (25), (26) and (27) or (29), (30) and (31), respectively, as follows:

$$\hat{R}(x) = p e^{-\left(\frac{\hat{\lambda}_1 x}{\hat{\beta}_1}\right)^{\hat{c}_1}} + (1-p) e^{-\left(\frac{\hat{\lambda}_2 x}{\hat{\beta}_2}\right)^{\hat{c}_2}}, \quad (32)$$

$$\hat{h}(x) = p \frac{\hat{c}_1 {\hat{\lambda}_1}^{\hat{c}_1}}{{\hat{\beta}_1}^{\hat{c}_1}} x^{\hat{c}_1 - 1} + (1-p) \frac{\hat{c}_2 {\hat{\lambda}_2}^{\hat{c}_2}}{{\hat{\beta}_2}^{\hat{c}_2}} x^{\hat{c}_2 - 1}, \quad (33)$$

$$\widehat{rh}(x) = p \frac{\frac{\hat{c}_1 {\hat{\lambda}_1}^{\hat{c}_1}}{{\hat{\beta}_1}^{\hat{c}_1}} x^{\hat{c}_1 - 1} e^{-\left(\frac{\hat{\lambda}_1 x}{\hat{\beta}_1}\right)^{\hat{c}_1}}}{1 - e^{-\left(\frac{\hat{\lambda}_1 x}{\hat{\beta}_1}\right)^{\hat{c}_1}}} + (1-p) \frac{\frac{\hat{c}_2 {\hat{\lambda}_2}^{\hat{c}_2}}{{\hat{\beta}_2}^{\hat{c}_2}} x^{\hat{c}_2 - 1} e^{-\left(\frac{\hat{\lambda}_2 x}{\hat{\beta}_2}\right)^{\hat{c}_2}}}{1 - e^{-\left(\frac{\hat{\lambda}_2 x}{\hat{\beta}_2}\right)^{\hat{c}_2}}}, \quad (34)$$

where $x > 0$, $\hat{c}_j$, $\hat{\lambda}_j$, $\hat{\beta}_j > 0$, $\hat{c}_j$, $\hat{\lambda}_j$ and $\hat{\beta}_j$ are the ML estimators of c_j, λ_j, β_j and $j = 1, 2$.

5 Simulation study

In this section, a simulation study is proceeded to evaluate the performance of the maximum likelihood estimators for each parameter of the new mixture distribution. The Monte Carlos simulation procedure is applied for full samples, Type I and Type II censoring samples, for different sample sizes ($n = 30$, 50,100 and, 150) and diffident mixture weights (p = 0.5 and 0.6). and diffident ratio of effective sizes (90%, 80% and 70%) Each simulation study is repeated for N = 1000. ML estimate, Bias and MSE are calculated for each situation and reported in Table (1), (2), (3), (4), (5) and (6).

In general, it is noted that most of the estimates of parameters by maximum likelihood estimation are accurate compared with the estimates of parameters by the maximum likelihood estimation with censored data Type I and Type II according to the resulted values of MSE.

Table 1. ML averages, Bias and MSE of MWE parameters based on Full Sample at $p = 0.5$

p	N	Results	Parameters			
			$c_1 = 15$	$\lambda_1 = 1.5$	$c_2 = 3$	$\lambda_2 = 0.5$
0.5	30	MLE	15.0491	1.5031	3.2571	0.5075
		Bias	0.0491	0.0031	0.2571	0.0075
		MSE	0.2458	0.0009	0.5127	0.0024
	50	MLE	15.0018	1.5023	3.1220	0.5070
		Bias	0.0018	0.0023	0.1220	0.0070
		MSE	0.2197	0.0005	0.2245	0.0016
	100	MLE	15.0113	1.5012	3.0652	0.5033
		Bias	0.0113	0.0012	0.0652	0.0033
		MSE	0.1285	0.0002	0.1197	0.0007
	150	MLE	15.0129	1.5002	3.0516	0.5022
		Bias	0.0129	0.0002	0.0516	0.0022
		MSE	0.0653	0.0002	0.0870	0.0005

Table 2. ML averages, Bias and MSE of MWE parameters based on Full Sample at $p = 0.6$.

p	N	Results	Parameters			
			$c_1 = 15$	$\lambda_1 = 1.5$	$c_2 = 3$	$\lambda_2 = 0.5$
0.6	30	MLE	15.0448	1.5029	3.2415	0.5099
		Bias	0.0448	0.0029	0.2415	0.0099
		MSE	0.2783	0.0007	0.4597	0.0031
	50	MLE	15.0015	1.5024	3.1391	0.5059
		Bias	0.0015	0.0024	0.1391	0.0059

			MSE	0.1960	0.0004	0.2416	0.0017
		100	MLE	15.0035	1.5009	3.0640	0.5046
			Bias	0.0035	0.0009	0.0640	0.0046
			MSE	0.0880	0.0002	0.1166	0.0009
		150	MLE	15.0020	1.5001	3.0492	0.5030
			Bias	0.0020	0.0001	0.0492	0.0030
			MSE	0.0607	0.0001	0.0817	0.0006

Table 3. ML averages, Bias and MSE of MWE parameters based on Type I Censored at $p = 0.5$.

p		N	Results	Parameters			
				$c_1 = 15$	$\lambda_1 = 1.5$	$c_2 = 3$	$\lambda_2 = 0.5$
0.5	90%	30	MLE	15.0035	1.4984	2.9986	0.5094
			Bias	0.0035	-0.0016	-0.0014	0.0094
			MSE	0.0200	0.0023	0.0065	0.0196
		50	MLE	14.9990	1.4995	2.9989	0.5042
			Bias	-9.8719e-04	-5.3945e-04	-1.1336e-03	4.2436e-03
			MSE	0.0016	0.0012	0.0021	0.0009
		100	MLE	14.9999	1.5009	2.9979	0.5053
			Bias	-0.0001	0.0009	-0.0021	0.0053
			MSE	0.0013	0.0012	0.0022	0.0009
		150	MLE	14.9979	1.499	2.9993	0.5051
			Bias	-2.0182e-03	-8.8570e-04	-7.0493e-04	5.0554e-03
			MSE	0.0014	0.0007	0.0012	0.0007
	80%	30	MLE	15.0005	1.4997	2.9972	0.5021
			Bias	0.0005	-0.0003	-0.0028	0.0021
			MSE	0.0027	0.0031	0.0023	0.0017
		50	MLE	14.9989	1.5002	2.9989	0.5005
			Bias	-0.0010	0.0002	-0.0011	0.0005
			MSE	0.0028	0.0011	0.0023	0.0008
		100	MLE	15.0010	1.5003	2.9988	0.5017
			Bias	0.0010	0.0003	-0.0013	0.0017
			MSE	0.0025	0.0018	0.0019	0.0007
		150	MLE	15.0022	1.4981	2.9985	0.5009
			Bias	0.0022	-0.0019	-0.0015	0.0009
			MSE	0.0019	0.0014	0.0016	0.0005
	70%	30	MLE	14.9961	1.5017	2.9982	0.4982
			Bias	-0.0039	0.0017	-0.0018	-0.0018
			MSE	0.0042	0.0039	0.0047	0.0017
		50	MLE	14.9993	1.4999	2.9965	0.4969

			Bias	-6.5548e-04	-5.0657e-05	-3.5099e-03	-3.1144e-03
			MSE	0.0025	0.0021	0.0031	0.0012
		100	MLE	14.9986	1.4972	2.9997	0.4978
			Bias	-0.0014	-0.0028	-0.0003	-0.0022
			MSE	0.0017	0.0011	0.0015	0.0008
		150	MLE	15.0004	1.4998	3.0012	0.4978
			Bias	0.0004	-0.0002	0.0012	-0.0022
			MSE	0.0018	0.0006	0.0014	0.0007

Table 4. ML averages, Bias and MSE of MWE parameters based on Type I Censored at $p = 0.6$.

p		N	Results	Parameters			
				$c_1 = 15$	$\lambda_1 = 1.5$	$c_2 = 3$	$\lambda_2 = 0.5$
0.6	90%	30	MLE	15.0008	1.4984	2.9999	0.5026
			Bias	0.0008	-0.0016	-0.0001	0.0028
			MSE	0.0032	0.0026	0.0036	0.0018
		50	MLE	15.0005	1.5001	2.9981	0.5046
			Bias	0.0005	0.0001	-0.0019	0.0046
			MSE	0.0029	0.00306	0.0050	0.0013
		100	MLE	14.9998	1.4996	2.9995	0.5036
			Bias	-0.0002	-0.0004	-0.0005	0.0036
			MSE	0.0011	0.0007	0.0029	0.0045
		150	MLE	15.0019	1.5019	2.9996	0.5019
			Bias	0.0019	0.0019	-0.0004	0.0019
			MSE	0.0009	0.0006	0.0008	0.0004
	80%	30	MLE	15.0053	1.4999	2.9997	0.5025
			Bias	0.0053	-0.0001	-0.0003	0.0025
			MSE	0.0220	0.0108	0.0080	0.0062
		50	MLE	14.9995	1.4976	2.9998	0.5004
			Bias	-0.0005	-0.0024	-0.0002	0.0004
			MSE	0.0018	0.0018	0.0018	0.0008
		100	MLE	15.0008	1.5004	2.9989	0.4998
			Bias	0.0008	0.0004	-0.0011	-0.0002
			MSE	0.0026	0.0009	0.0014	0.0005
		150	MLE	15.00009	1.4990	2.9986	0.5001
			Bias	8.4979e-05	-9.9213e-04	-1.3512e-03	5.7772e-05
			MSE	0.0017	0.0016	0.0020	0.0007
	70%	30	MLE	15.0008	1.4977	2.9955	0.5027
			Bias	0.0008	-0.0023	-0.0045	0.0027
			MSE	0.0059	0.0049	0.0118	0.0091

		50	MLE	14.9993	1.5013	3.0029	0.4971
			Bias	-0.0007	0.0013	0.0029	-0.0029
			MSE	0.0035	0.0086	0.0099	0.0014
		100	MLE	14.9981	1.5005	2.9995	0.4961
			Bias	-0.0019	0.0005	-0.0005	-0.0039
			MSE	0.0021	0.0014	0.0017	0.0008
		150	MLE	15.0011	1.4979	3.0006	0.4978
			Bias	0.0011	-0.0021	0.0006	-0.0022
			MSE	0.0009	0.0008	0.0011	0.0005

Table 5. ML averages, Bias and MSE of MWE parameters based on Type II Censored at $p = 0.5$.

p		N	Results	Parameters			
				$c_1 = 15$	$\lambda_1 = 1.5$	$c_2 = 3$	$\lambda_2 = 0.5$
0.5	90%	30	MLE	15.0468	1.5031	3.3146	0.5126
			Bias	0.0468	0.0031	0.3146	0.0126
			MSE	0.2392	0.0009	0.6042	0.0031
		50	MLE	15.0247	1.5024	3.1965	0.5064
			Bias	0.0247	0.0024	0.1965	0.0064
			MSE	0.2195	0.0005	0.3317	0.0016
		100	MLE	14.9984	1.5012	3.0864	0.5044
			Bias	-0.0016	0.0012	0.0864	0.0044
			MSE	0.0974	0.0002	0.1567	0.0008
		150	MLE	15.0030	1.5002	3.0637	0.5030
			Bias	0.0030	0.0002	0.0637	0.0030
			MSE	0.0443	0.0002	0.1052	0.0005
	80%	30	MLE	15.0331	1.5030	3.3339	0.5233
			Bias	0.0331	0.0030	0.3339	0.0233
			MSE	0.2719	0.0009	0.5828	0.0061
		50	MLE	15.0014	1.5025	3.2351	0.5109
			Bias	0.0014	0.0025	0.2351	0.0109
			MSE	0.2548	0.0005	0.3992	0.0027
		100	MLE	14.9940	1.5012	3.0779	0.5069
			Bias	-0.0060	0.0012	0.0779	0.0069
			MSE	0.0618	0.0002	0.1345	0.0012
		150	MLE	15.0024	1.5002	3.0595	0.5039
			Bias	0.0024	0.0002	0.0595	0.0039
			MSE	0.0524	0.0002	0.1066	0.0008
	70%	30	MLE	15.0763	1.5030	3.4335	0.5617
			Bias	0.0763	0.0030	0.4335	0.0617
			MSE	0.4015	0.0009	0.7830	0.0274
		50	MLE	15.0011	1.5026	3.2327	0.5257
			Bias	0.0011	0.0026	0.2327	0.0257

			MSE	0.1959	0.0005	0.3716	0.0083
		100	MLE	14.9886	1.5013	3.0751	0.5123
			Bias	-0.0114	0.0013	0.0751	0.0123
			MSE	0.0704	0.0002	0.1228	0.0023
		150	MLE	15.0123	1.5003	3.0677	0.5069
			Bias	0.0123	0.0003	0.0677	0.0069
			MSE	0.0595	0.0002	0.1100	0.0014

Table 6. ML averages, Bias and MSE of MWE parameters based on Type II Censored at

$\mathbf{p = 0.6}$.

p		N	Results	Parameters			
				$c_1 = 15$	$\lambda_1 = 1.5$	$c_2 = 3$	$\lambda_2 = 0.5$
0.6	90%	30	MLE	15.0636	1.5029	3.26316	0.5192
			Bias	0.0636	0.0029	0.2631	0.0192
			MSE	0.2243	0.0007	0.4572	0.0056
		50	MLE	15.0421	1.5024	3.1812	0.5091
			Bias	0.0421	0.0024	0.1812	0.0091
			MSE	0.2832	0.0004	0.3030	0.0023
		100	MLE	15.0087	1.5009	3.0687	0.5068
			Bias	0.0087	0.0009	0.0687	0.0068
			MSE	0.0434	0.0002	0.1136	0.0012
		150	MLE	15.0041	1.5002	3.0470	0.5040
			Bias	0.0041	0.0002	0.0470	0.0040
			MSE	0.0372	0.0001	0.0789	0.0007
	80%	30	MLE	15.0658	1.5029	3.3378	0.5532
			Bias	0.0658	0.0029	0.3378	0.0532
			MSE	0.2743	0.0007	0.5779	0.0220
		50	MLE	15.0854	1.5024	3.2663	0.5270
			Bias	0.0854	0.0024	0.2663	0.0270
			MSE	0.3373	0.0004	0.4731	0.0078
		100	MLE	15.0065	1.5010	3.0696	0.5122
			Bias	0.0065	0.0010	0.0696	0.0122
			MSE	0.0499	0.0002	0.0873	0.0021
		150	MLE	15.0013	1.5002	3.0430	0.5064
			Bias	0.0013	0.0002	0.0430	0.0064
			MSE	0.05682	0.00011	0.0835	0.0013
	70%	30	MLE	15.0954	1.5021	3.3493	0.6215
			Bias	0.0954	0.0021	0.3493	0.1215
			MSE	0.3082	0.0008	0.4777	0.0537
		50	MLE	15.0812	1.5021	3.2511	0.5845
			Bias	0.0812	0.0021	0.2511	0.0845
			MSE	0.3786	0.0004	0.3195	0.0329
		100	MLE	15.0095	1.5011	3.1110	0.5422

			Bias	0.0095	0.0011	0.1110	0.0422
			MSE	0.0988	0.0002	0.1179	0.0146
		150	MLE	15.0081	1.5003	3.0645	0.5217
			Bias	0.0081	0.0003	0.0645	0.0217
			MSE	0.0813	0.0001	0.0864	0.0060

6 Concluding remarks

- It is noticed, from Tables (1) and (2) that the ML averages with full samples are very close to the initial values of the parameters as the sample size increases. Also, Bias's and MSEs are decreasing when the sample size is increasing. This is indicative of the fact that the estimates are consistent and approaches the population parameter values as the sample size increases.

- It is noticed, from Tables (3) and (4), that the ML averages with Type-I censored samples are very close to the initial values of the parameters as the sample size increases. Also, Bias's and MSEs are decreasing when the sample size is increasing. This is indicative of the fact that the estimates are consistent and approaches the population parameter values as the sample size increases.

- It is noticed, from Tables (5) and (6), that the ML averages with Type-II censored samples are very close to the initial values of the parameters as the sample size increases. Also, Bias's and MSEs are decreasing when the sample size is increasing. This is indicative of the fact that the estimates are consistent and approaches the population parameter values as the sample size increases.
- It is noticed, from Table (1), (2), (3), (4), (5) and (6) that the parameters of the MWE model are estimated by the maximum likelihood estimation method with full samples, Type-I and Type-II censored samples and it shows that as the percent of censored data Type I and Type II increases, the estimates become more accurate which confirms that the most accurate estimator is the maximum likelihood estimation with full sample.

7 General Conclusion

In this study, the MWE distribution was introduced based on mixture approach in order to provide flexibility in fitting different types of data. General statistical properties were obtained. The maximum likelihood estimation method was employed for estimating the parameters of the proposed distribution based on complete samples, Type-I and Type-II censored samples. The performances of these MLEs were tested through simulation studies. The ML estimation method produced good estimators for the parameters of the MWE. These estimates are consistent since Bias's and MSEs are small and decreasing when the sample size is increasing. This study indicates that the introduced distribution MWR can offer the best fit for mixture data in different areas.

References

Elshahat, A. T. and Mahmoud, A. A. (2016). A Study on the Mixture of Exponentiated Weibull Distribution, Pakistan Journal of Statistics and Operation Research, vol. 12:709-737.

Figueiredo, M. A. T. and Jain, A. K. (2002) Unsupervised Learning of Finite Mixture Models, IEEE Transactions on Pattern Analysis & Machine Intelligence,vol. 24:381-396.

Franco, M., Vivo, J., and Balakrishnan, N. (2011) Reliability Properties of Generalized Mixtures of Weibull Distributions with a Common Shape Parameter, Journal of Statistical Planning and Inference, vol. 141:2600-2613.

Huang, T., Peng, H., and Zhang, K. (2017) Model Selection for Gaussian Mixture Models, Statistica Sinica, 147-169.

McLachlan, G. J., Lee, S. X., and Rathnayake, S. I. (2019) Finite mixture Models, Annual Review of Statistics and Its Application, vol. 6: 355-378.

Newcomb, S. (1886) A Generalized Theory of the Combination of Observations so as to Obtain the Best Result, American journal of Mathematics, vol. 8: 343-366.

Pearson, K. (1894) Contributions to the Mathematical Theory of Evolution, Philosophical Transactions of the Royal Society of London. A, vol. 185:71-110.

Qutb, N., Adham, S. A., and Dandeni, N. K. (2016) Estimation of a Mixture of Two Weibull Distributions under Generalized Order Statistics.

Razali, A. M., & Al-Wakeel, A. A. (2013) Mixture Weibull Distributions for Fitting Failure Times Data, Applied Mathematics and Computation, vol. 219: 11358-11364.

Teamah, A. E. A., Elbanna, A. A., and Gemeay, A. M. (2020) Fréchet-Weibull Mixture Distribution: Properties And Applications, Applied Mathematical Sciences, vol.14: 75-86.

Zhang, Q., Hua, C., & Xu, G. (2014) A Mixture Weibull Proportional Hazard Model for Mechanical System Failure Prediction Utilising Lifetime and Monitoring Data, Mechanical Systems and Signal Processing, vol. 43: 103-112.

Zong, B., Song, Q., Min, M. R., Cheng, W., Lumezanu, C., Cho, D., & Chen, H. (2018, February) Deep Autoencoding Gaussian Mixture Model for Unsupervised Anomaly Detection, In International Conference on Learning Representations.